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Uniqueness and value-sharing of meromorphic functions

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Abstract

In this paper, we study the uniqueness of meromorphic functions and prove the following theorem: Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, n, k two positive integers with $n > 3k + 8$. If $[f^n(z)]^{(k)}$ and $[g^n(z)]^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$ or $f(z) = tg(z)$ for a constant t such that $t^n = 1$. Our results improves the results of Fang [M.L. Fang, Uniqueness and value-sharing of entire functions, Comput. Math. Appl. 44 (2002) 823–831. [7]], Fang and Hong [M.L. Fang, W. Hong, A unicity theorem for entire functions concerning differential polynomials, Indian J. Pure Appl. Math. 32 (9) (2001) 1343–1348. [8]] and Lin and Yi [W.-C. Lin, H.-X. Yi, Uniqueness theorems for meromorphic function, Indian J. Pure Appl. Math. 32 (9) (2004) 121–132. [9]].

Keywords: Meromorphic functions; Entire functions; Shared values; Differential polynomials; Uniqueness

1. Introduction

Let $f(z)$ be a non-constant meromorphic function in the whole complex plane. We shall use the following standard notations of the value distribution theory:

$$T(r, f), m(r, f), N(r, f), \bar{N}(r, f), \ldots$$

(see Hayman [1], Yang [2] and Yi and Yang [15]). We denote by $S(r, f)$ any quantity satisfying

$$S(r, f) = o(T(r, f)),$$

as $r \to +\infty$, possibly outside of a set with finite measure. For any constant 'a' we define

$$\Theta(a, f) = 1 - \overline{\lim_{r\to\infty}} \frac{\bar{N}\left(r, \frac{1}{f-a}\right)}{T(r, f)}.$$

Let 'a' be a finite complex number and k a positive integer. We denote by $N_{k)}(r, \frac{1}{f-a})$ the counting function for zeros of $f(z) - a$ with multiplicity $\leq k$, and by $\bar{N}_{k)}(r, \frac{1}{f-a})$ the corresponding one for which multiplicity is not

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counted. Let $N_{(k}(r, \frac{1}{f-a})$ be the counting function for zeros of $f(z) - a$ with multiplicity at least k and $\bar{N}_{(k}(r, \frac{1}{f-a})$ the corresponding one for which multiplicity is not counted.

Set

$$N_k\left(r, \frac{1}{f-a}\right) = \bar{N}\left(r, \frac{1}{f-a}\right) + \bar{N}_{(2}\left(r, \frac{1}{f-a}\right) + \cdots + \bar{N}_{(k}\left(r, \frac{1}{f-a}\right).$$

We define

$$\delta_k(a, f) = 1 - \overline{\lim_{r\to\infty}} \frac{N_k\left(r, \frac{1}{f-a}\right)}{T(r, f)}.$$

Let $g(z)$ be a meromorphic function. If $f(z) - a$ and $g(z) - a$, assume the same zeros with the same multiplicities then we say that $f(z)$ and $g(z)$ share the value 'a' CM, where 'a' is a complex number.

Hayman [3] and Clunie [4] proved the following result

Theorem A. *Let $f(z)$ be a transcendental entire function, $n \geq 1$ a positive integer, then $f^n f' = 1$ has infinitely many solutions.*

Fang and Hua [5] and Yang and Hua [6] obtained a unicity theorem corresponding to the above result.

Theorem B. *Let $f(z)$ and $g(z)$ be two non-constant entire functions, $n \geq 6$ a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1c_2)^{n+1}c^2 = -1$ or $f(z) = tg(z)$ for a constant t such that $t^{n+1} = 1$.*

Hennekemper [10], Chen [11] and Wang [12,13] extended Theorem A by proving the following theorem.

Theorem C. *Let $f(z)$ be a transcendental entire function, n, k two positive integers with $n \geq k + 1$. Then $(f^n(z))^{(k)} = 1$ has infinitely many solutions.*

Fang [7] obtained a unicity theorem corresponding to Theorem C.

Theorem D. *Let $f(z)$ and $g(z)$ be two non-constant entire functions, and let n, k be two positive integers with $n > 2k + 4$. If $[f^n(z)]^{(k)}$ and $[g^n(z)]^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1c_2)^n (nc)^{2k} = 1$ or $f(z) = tg(z)$ for a constant t such that $t^n = 1$.*

In this paper, we extend Theorems C and D to meromorphic functions by proving

Theorem 1. *Let $f(z)$ be a transcendental meromorphic function and let n, k be two positive integers with $n \geq k+3$. Then $(f^n(z))^{(k)} = 1$ has infinitely many solutions.*

In view of Theorem 1, Theorem D naturally motivates us to the following theorem.

Theorem 2. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, and let n, k be two positive integers with $n > 3k+8$. If $[f^n(z)]^{(k)}$ and $[g^n(z)]^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1c_2)^n (nc)^{2k} = 1$ or $f(z) = tg(z)$ for a constant t such that $t^n = 1$.*

Using the same argument as in [3], we prove the following result for transcendental meromorphic functions.

Theorem 3. *Let $f(z)$ be a transcendental meromorphic function, n, k two positive integers with $n \geq k + 3$. Then $[f^n(f-1)]^{(k)} = 1$ has infinitely many solutions.*

In this paper, we also obtain a corresponding unicity theorem to Theorem 3.

Theorem 4. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions satisfying $\Theta(\infty, f) > \frac{3}{n+1}$ and let n, k be two positive integers with $n \geq 3k+13$. If $[f^n(z)(f(z)-1)]^{(k)}$ and $[g^n(z)(g(z)-1)]^{(k)}$ share 1 CM, then $f(z) \equiv g(z)$.*

2. Some lemmas

For the proof of our results we need the following lemmas.

Lemma 1 (*[1]*). *Let $f(z)$ be a non-constant meromorphic function, k a positive integer, and let c be a non-zero finite complex number. Then*

$$T(r,f) \le \bar{N}(r,f) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^{(k)}-c}\right) - N\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f) \tag{2.1}$$

$$\le \bar{N}(r,f) + N_{k+1}\left(r,\frac{1}{f}\right) + \bar{N}\left(r,\frac{1}{f^{(k)}-c}\right) - N_0\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f), \tag{2.2}$$

where $N_0(r,\frac{1}{f^{(k+1)}})$ is the counting function which only counts those points such that $f^{(k+1)}=0$ but $f(f^{(k)}-c)\ne 0$.

Lemma 2 (*[1]*). *Let $f(z)$ be a meromorphic function and "a" is a finite complex number. Then*

(i) $T(r,\frac{1}{f-a}) = T(r,f) + O(1)$,

(ii) $m(r,\frac{f^{(k)}}{f^{(l)}}) = S(r,f)$, *for* $k > l \ge 0$,

(iii) $T(r,f) \le \bar{N}(r,f) + \bar{N}(r,\frac{1}{f-a_1(z)}) + \bar{N}(r,\frac{1}{f-a_2(z)}) + S(r,f)$,

where $a_1(z), a_2(z)$ are two meromorphic functions such that $T(r,a_i) = S(r,f)$, $(i=1,2)$.

Lemma 3 (*[1]*). *Let $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, where a_n $(\ne 0), a_{n-1}, \ldots, a_0$ are constants. If $f(z)$ is a non-constant meromorphic function, then*

$$T(r,p(f)) = nT(r,f) + S(r,f).$$

Lemma 4 (*[14]*). *Let $f(z)$ be a non-constant entire function, and let $k \ge 2$ be a positive integer. If $f(z)f^{(k)}(z) \ne 0$, then $f = e^{az+b}$, where $a \ne 0$, b are constants.*

We now prove the following lemma which plays a cardinal role in the proof of Theorems 1 and 2.

Lemma 5. *Let $f(z)$ and $g(z)$ be two meromorphic functions, and let k be a positive integer. If $f^{(k)}$ and $g^{(k)}$ share the value 1 CM and*

$$\Delta = \left[(k+2)\Theta(\infty,f) + 2\Theta(\infty,g) + \Theta(0,f) + \Theta(0,g) + \delta_{k+1}(0,f) + \delta_{k+1}(0,g)\right] > k+7, \tag{2.3}$$

then either $f^{(k)}g^{(k)} \equiv 1$ or $f \equiv g$.

Proof. Let

$$\Phi(z) = \frac{f^{(k+2)}}{f^{(k+1)}} - 2\frac{f^{(k+1)}}{f^{(k)}-1} - \frac{g^{(k+2)}}{g^{(k+1)}} + 2\frac{g^{(k+1)}}{g^{(k)}-1}. \tag{2.4}$$

Suppose

$$\Phi(z) \not\equiv 0.$$

If z_0 is a common simple 1-point of $f^{(k)}(z)$ and $g^{(k)}(z)$, substituting their Taylor series at z_0 into (2.4), we see that z_0 is a zero of $\Phi(z)$.

Thus, we have

$$N_{1)}\left(r,\frac{1}{f^{(k)}-1}\right) = N_{1)}\left(r,\frac{1}{g^{(k)}-1}\right) \le \bar{N}\left(r,\frac{1}{\Phi}\right) \le T(r,\Phi) + O(1) \le N(r,\Phi) + S(r,f) + S(r,g), \tag{2.5}$$

here $N_{1)}(r,\frac{1}{f^{(k)}-1})$ is the counting function which only counts those points such that $f^{(k)}-1=0$ but $f^{(k+1)}\ne 0$.

Our assumptions are that $\Phi(z)$ has simple poles only at zeros of $f^{(k+1)}$ and $g^{(k+1)}$ and poles of f and g. Thus, we deduce from (2.4) that

$$N(r,\Phi) \le \bar{N}(r,f) + \bar{N}(r,g) + \bar{N}\left(r,\frac{1}{f}\right) + \bar{N}\left(r,\frac{1}{g}\right) + N_0\left(r,\frac{1}{f^{(k+1)}}\right) + N_0\left(r,\frac{1}{f^{(k+1)}}\right), \tag{2.6}$$

here $N_0(r,\frac{1}{f^{(k+1)}})$ has the same meaning as in Lemma 1. Obviously,

$$\bar{N}\left(r,\frac{1}{f^{(k)}-1}\right) + \bar{N}\left(r,\frac{1}{g^{(k)}-1}\right) = 2\bar{N}\left(r,\frac{1}{f^{(k)}-1}\right) \le N_{1)}\left(r,\frac{1}{f^{(k)}-1}\right) + N\left(r,\frac{1}{f^{(k)}-1}\right). \tag{2.7}$$

From Lemma 1, we have

$$T(r,f) \le \bar{N}(r,f) + N_{k+1}\left(r,\frac{1}{f}\right) + \bar{N}\left(r,\frac{1}{f^{(k)}-c}\right) - N_0\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f), \tag{2.8}$$

$$T(r,g) \le \bar{N}(r,g) + N_{k+1}\left(r,\frac{1}{g}\right) + \bar{N}\left(r,\frac{1}{g^{(k)}-c}\right) - N_0\left(r,\frac{1}{g^{(k+1)}}\right) + S(r,g). \tag{2.9}$$

Thus, we deduce from (2.5)–(2.9) that

$$\begin{aligned} T(r,f) + T(r,g) \le\ & 2\bar{N}(r,f) + 2\bar{N}(r,g) + N_{k+1}\left(r,\frac{1}{f}\right) + N_{k+1}\left(r,\frac{1}{g}\right) + \bar{N}\left(r,\frac{1}{f}\right) \\ & + \bar{N}\left(r,\frac{1}{g}\right) + N\left(r,\frac{1}{f^{(k)}-1}\right) + S(r,f) + S(r,g). \end{aligned} \tag{2.10}$$

Since

$$\begin{aligned} N\left(r,\frac{1}{f^{(k)}-1}\right) &\le T(r,f^{(k)}) = m(r,f^{(k)}) + N(r,f^{(k)}) \\ &\le m(r,f) + m\left(r,\frac{f^{(k)}}{f}\right) + N(r,f) + k\bar{N}(r,f) \\ &\le T(r,f) + k\bar{N}(r,f) + S(r,f). \end{aligned}$$

We obtain from (2.10) that

$$\begin{aligned} T(r,g) \le\ & (k+2)\bar{N}(r,f) + 2\bar{N}(r,g) + N_{k+1}\left(r,\frac{1}{f}\right) + N_{k+1}\left(r,\frac{1}{g}\right) \\ & + \bar{N}\left(r,\frac{1}{f}\right) + \bar{N}\left(r,\frac{1}{g}\right) + S(r,f) + S(r,g). \end{aligned}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f) \le T(r,g)$ for $r \in I$. Hence

$$T(r,g) \le \left[\begin{matrix} 1-\delta_{k+1}(0,f) + 1 - \delta_{k+1}(0,g) + 1 - \Theta(0,f) \\ +1 - \Theta(0,g) + (k+2)[1-\Theta(\infty,f)] + 2[1-\Theta(\infty,g)] + \varepsilon \end{matrix}\right] T(r,g) + S(r,g), \tag{2.11}$$

for $r \in I$ and $0 < \varepsilon < \Delta - (k+7)$, that is $\{\Delta - (k+7) - \varepsilon\}T(r,g) \le S(r,g)$, i.e.

$$\Delta - (k+7) \le 0$$

i.e.

$$\Delta \le k+7$$

which is a contradiction to our hypothesis $\Delta > k+7$ from (2.3).

Hence, we get $\Phi(z) \equiv 0$; that is,

$$\frac{f^{(k+2)}}{f^{(k+1)}} - 2\frac{f^{(k+1)}}{f^{(k)}-1} \equiv \frac{g^{(k+2)}}{g^{(k+1)}} - 2\frac{g^{(k+1)}}{g^{(k)}-1}. \tag{2.12}$$

Integrating this equation, we get

$$\log f^{(k+1)} - 2\log(f^{(k)} - 1) = \log g^{(k+1)} - 2\log(g^{(k)} - 1) + \log a,$$

where a is constant and $a \neq 0$.

That is

$$\log \frac{f^{(k+1)}}{(f^{(k)} - 1)^2} = \log \frac{ag^{(k+1)}}{(g^{(k)} - 1)^2},$$

i.e.

$$\frac{f^{(k+1)}}{(f^{(k)} - 1)^2} = \frac{ag^{(k+1)}}{(g^{(k)} - 1)^2}.$$

Again integrating the above equation, we get

$$-\frac{1}{f^{(k)} - 1} = -\frac{a}{g^{(k)} - 1} - b,$$

where b is a constant.

Solving the above equation, we get

$$\frac{1}{f^{(k)} - 1} = \frac{bg^{(k)} + a - b}{g^{(k)} - 1}, \tag{2.13}$$

where a, b are two constants and $a \neq 0$.

Next, we consider three cases

Case 1: $a = b$. From (2.13),

(i) If $b = -1$, then

$$f^{(k)}(z)g^{(k)}(z) \equiv 1.$$

(ii) If $b \neq -1$, then

$$\frac{1}{f^{(k)} - 1} = \frac{bg^{(k)}}{g^{(k)} - 1}$$

$$\frac{1}{f^{(k)}} = \frac{bg^{(k)}}{(1+b)g^{(k)} - 1}. \tag{2.14}$$

We can write

$$\bar{N}\left[r, \left(\frac{1}{g^{(k)} - (1/(1+b))}\right)\right] \leq \bar{N}\left[r, \left(\frac{g^{(k)}}{g^{(k)} - (1/(1+b))}\right)\right]. \tag{2.15}$$

From (2.14), we have

$$\bar{N}\left[r, \left(\frac{g^{(k)}}{g^{(k)} - (1/(1+b))}\right)\right] = \bar{N}\left(r, \frac{1}{f^{(k)}}\right). \tag{2.16}$$

From (2.15) and (2.16), we get

$$\bar{N}\left[r, \left(\frac{1}{g^{(k)} - (1/(1+b))}\right)\right] \leq \bar{N}\left(r, \frac{1}{f^{(k)}}\right). \tag{2.17}$$

By Lemma 2, we obtain the following inequality

$$\begin{aligned}\bar{N}\left(r, \frac{1}{f^{(k)}}\right) &\leq \bar{N}\left(r, \frac{f}{f^{(k)}}\right) + \bar{N}\left(r, \frac{1}{f}\right) \leq T\left(r, \frac{f}{f^{(k)}}\right) + \bar{N}\left(r, \frac{1}{f}\right)\\ &\leq T\left(r, \frac{f^{(k)}}{f}\right) + \bar{N}\left(r, \frac{1}{f}\right) + S(r, f)\end{aligned}$$

$$\leq N\left(r, \frac{f^{(k)}}{f}\right) + m\left(r, \frac{f^{(k)}}{f}\right) + \bar{N}\left(r, \frac{1}{f}\right) + S(r, f)$$
$$\leq k\bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f}\right) + S(r, f)$$
$$\leq (k+2)\bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f}\right) + S(r, f).$$

Therefore

$$\bar{N}\left(r, \frac{1}{f^{(k)}}\right) \leq (k+2)\bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f}\right) + S(r, f). \tag{2.18}$$

From (2.17) and (2.18), we get

$$\bar{N}\left[r, \left(\frac{1}{g^{(k)} - (1/(1+b))}\right)\right] \leq \bar{N}\left(r, \frac{1}{f^{(k)}}\right) \leq (k+2)\bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f}\right) + S(r, f).$$

From Lemma 1, we have

$$\begin{aligned} T(r, g) &\leq \bar{N}(r, g) + N_{k+1}\left(r, \frac{1}{g}\right) + \bar{N}\left(r, \frac{1}{g^{(k)} - c}\right) - N_0\left(r, \frac{1}{g^{(k+1)}}\right) \\ &\leq \bar{N}(r, g) + N_{k+1}\left(r, \frac{1}{g}\right) + \bar{N}\left(r, \frac{1}{g^{(k)} - (1/(1+b))}\right) + S(r, g) \quad \text{since } c = 1/(1+b) \neq 0 \\ &\leq \bar{N}(r, g) + N_{k+1}\left(r, \frac{1}{g}\right) + (k+2)\bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f}\right) + S(r, f) + S(r, g) \\ &\leq (k+2)\bar{N}(r, f) + 2\bar{N}(r, g) + N_{k+1}\left(r, \frac{1}{f}\right) + N_{k+1}\left(r, \frac{1}{g}\right) + \bar{N}\left(r, \frac{1}{f}\right) \\ &\quad + \bar{N}\left(r, \frac{1}{g}\right) + S(r, f) + S(r, g). \end{aligned}$$

That is

$$T(r, g) \leq (k + 8 - \Delta)T(r, g) + S(r, g),$$

for $r \in I$ and r sufficiently large.

That is

$$(\Delta - k - 7)T(r, g) \leq S(r, g).$$

Hence, by (2.3), we deduce that $T(r, g) \leq S(r, g)$, a contradiction.

Case 2: $b \neq 0$ and $a \neq b$. Then from (2.13),

(i) If $b = -1$, we obtain

$$f^{(k)} = \frac{a}{-g^{(k)} + a + 1}.$$

Therefore

$$\bar{N}\left(r, \frac{1}{g^{(k)} - (1+a)}\right) = \bar{N}(r, f^{(k)}) = \bar{N}(r, f).$$

From Lemma 1, we have

$$\begin{aligned} T(r, g) &\leq \bar{N}(r, g) + N_{k+1}\left(r, \frac{1}{g}\right) + \bar{N}\left(r, \frac{1}{g^{(k)} - (1+a)}\right) + S(r, g) \quad \text{since } c = 1 + a \neq 0 \\ &\leq \bar{N}(r, g) + N_{k+1}\left(r, \frac{1}{g}\right) + \bar{N}(r, f) + S(r, g) \\ &\leq (k+2)\bar{N}(r, f) + 2\bar{N}(r, g) + N_{k+1}\left(r, \frac{1}{f}\right) + N_{k+1}\left(r, \frac{1}{g}\right) + \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{g}\right) + S(r, g). \end{aligned}$$

Using the argument as in case 1, we get a contradiction.

(ii) If $b \neq -1$, we obtain that

$$f^{(k)} - (1 + 1/b) = \frac{-a}{b^2 \left[g^{(k)} + \frac{a-b}{b}\right]}.$$

Therefore

$$\bar{N}\left[r, \left(\frac{1}{g^{(k)} + \frac{a-b}{b}}\right)\right] = \bar{N}(r, f^{(k)} - (1 + 1/b)) = \bar{N}(r, f^{(k)}) = \bar{N}(r, f).$$

From Lemma 1, we have

$$\begin{aligned} T(r, g) &\leq \bar{N}(r, g) + N_{k+1}\left(r, \frac{1}{g}\right) + \bar{N}\left(r, \frac{1}{g^{(k)} + \frac{a-b}{b}}\right) - N_0\left(r, \frac{1}{g^{(k+1)}}\right) + S(r, g), \\ &\leq \bar{N}(r, g) + N_{k+1}\left(r, \frac{1}{g}\right) + \bar{N}(r, f) + S(r, g), \\ &\leq (k+2)\bar{N}(r, f) + 2\bar{N}(r, g) + N_{k+1}\left(r, \frac{1}{f}\right) + N_{k+1}\left(r, \frac{1}{g}\right) + \bar{N}\left(r, \frac{1}{f}\right) \\ &\quad + \bar{N}\left(r, \frac{1}{g}\right) + S(r, f) + S(r, g). \end{aligned}$$

Using the argument as in case 1, we get a contradiction.

Case 3: $b = 0$. From (2.13), we obtain

$$f = \frac{1}{a} g + p(z), \tag{2.19}$$

where $p(z)$ is a polynomial. If $p(z) \not\equiv 0$, then by Lemma 2, we have

$$\begin{aligned} T(r, f) &\leq \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{f - p}\right) + S(r, f) \\ &\leq \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{g}\right) + S(r, f). \end{aligned} \tag{2.20}$$

From (2.19), we obtain

$$T(r, f) = T(r, g) + S(r, f).$$

Hence, substituting this into (2.20), we get

$$T(r, f) \leq \{3 - [\Theta(\infty, f) + \Theta(0, f) + \Theta(0, g)] + \varepsilon\}\, T(r, f) + S(r, f), \tag{2.21}$$

where

$$0 < \varepsilon < 1 - \delta_{k+1}(0, f) + 1 - \delta_{k+1}(0, g) + 2[1 - \Theta(\infty, g)] + (k+1)[1 - \Theta(\infty, f)].$$

Therefore

$$T(r, f) \leq \{k + 8 - \Delta\}\, T(r, f) + S(r, f).$$

That is

$$[\Delta - k - 7] T(r, f) \leq S(r, f).$$

Hence, by (2.3), we deduce that $T(r, f) \leq S(r, f)$, a contradiction.

Therefore, we deduce that $p(z) \equiv 0$, that is,

$$f = \frac{1}{a} g. \tag{2.22}$$

If $a \neq 1$, then $f^{(k)}$ and $g^{(k)}$ sharing the value 1 CM, we deduce from (2.22) that $g^{(k)} \neq 1$. That is $\bar{N}(r, \frac{1}{g^{(k)}-1}) = 0$. Next, we can deduce a contradiction as in case 3. Thus, we get that $a = 1$, that is, $f \equiv g$. Thus proof of Lemma 5 is completed.

Lemma 6 ([1]). *Suppose that $f_1(z), f_2(z), \ldots, f_n(z) (n \geq 2)$ are meromorphic functions and $g_1(z), g_2(z), \ldots, g_n(z)$ are entire functions satisfying the following conditions*

(i) $\sum_{j=1}^{n} f_j(z) e^{g_j(z)} \equiv 0$,
(ii) $g_j(z) - g_k(z)$ *are not constants for* $1 \leq j < k \leq n$,
(iii) *For* $1 \leq j \leq n, 1 \leq h < k \leq n, T(r, f_j) = o\{T(r, e^{g_h - g_k})\}(r \to \infty, r \notin E)$.

Then $f_j(z) \equiv 0$ $(j = 1, 2, \ldots, n)$.

Remark 5. In fact, we only need to assume that the growth condition of Lemma 6 holds on a set of values of infinite linear measure.

3. Proof of Theorem 1

From Lemma 3 and (2.1), we have

$$\begin{aligned} nT(r, f) &= T\left(r, f^n(z)\right) \\ &\leq N\left(r, \frac{1}{f^n(z)}\right) + N\left(r, \frac{1}{[f^n(z)]^{(k)} - 1}\right) + \bar{N}\left(r, f^n(z)\right) - N\left(r, \frac{1}{(f^n(z))^{(k+1)}}\right) + S(r, f) \\ &\leq \bar{N}(r, f) + nN\left(r, \frac{1}{f}\right) - (n-k-1)N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{(f^n(z))^{(k)} - 1}\right) + S(r, f) \\ &\leq \bar{N}(r, f) + (k+1)N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{[f^n(z)]^{(k)} - 1}\right) + S(r, f) \\ &\leq (k+2)T(r, f) + N\left(r, \frac{1}{[f^n(z)]^{(k)} - 1}\right) + S(r, f). \end{aligned}$$

Therefore

$$(n-k-2)T(r, f) \leq N\left(r, \frac{1}{[f^n(z)]^{(k)} - 1}\right) + S(r, f). \tag{3.1}$$

Hence, we deduce from (3.1) and $n \geq k+3$ that $(f^n(z))^{(k)} - 1$ has infinitely many solutions.

4. Proof of Theorem 2

Consider $F(z) = f^n(z)$ and $G(z) = g^n(z)$. We have

$$\Delta = \left[(k+2)\Theta(\infty, F) + 2\Theta(\infty, G) + \Theta(0, F) + \Theta(0, G) + \delta_{k+1}(0, F) + \delta_{k+1}(0, G)\right]. \tag{4.1}$$

Consider

$$\begin{aligned} \Theta(0, F) &= 1 - \overline{\lim_{r\to\infty}} \frac{\bar{N}\left(r, \frac{1}{F}\right)}{T(r, F)} = 1 - \overline{\lim_{r\to\infty}} \frac{\bar{N}\left(r, \frac{1}{f^n}\right)}{nT(r, f)} \\ &= 1 - \overline{\lim_{r\to\infty}} \frac{\bar{N}\left(r, \frac{1}{f}\right)}{nT(r, f)} \geq 1 - \overline{\lim_{r\to\infty}} \frac{T(r, f)}{nT(r, f)}, \end{aligned}$$

i.e.

$$\Theta(0, F) \geq \frac{n-1}{n}. \tag{4.2}$$

Similarly

$$\Theta(0, G) \geq \frac{n-1}{n}. \tag{4.3}$$

Consider

$$\begin{aligned}\Theta(\infty, F) &= 1 - \overline{\lim_{r\to\infty}} \frac{\bar{N}(r, F)}{T(r, F)} = 1 - \overline{\lim_{r\to\infty}} \frac{\bar{N}(r, f^n)}{nT(r, f)}\\ &= 1 - \overline{\lim_{r\to\infty}} \frac{\bar{N}(r, f)}{nT(r, f)} \geq 1 - \overline{\lim_{r\to\infty}} \frac{T(r, f)}{nT(r, f)},\end{aligned}$$

i.e.

$$\Theta(\infty, F) \geq \frac{n-1}{n}. \tag{4.4}$$

Similarly

$$\Theta(\infty, G) \geq \frac{n-1}{n}. \tag{4.5}$$

Next, we have

$$N_k\left(r, \frac{1}{f-a}\right) = \bar{N}\left(r, \frac{1}{f-a}\right) + \bar{N}_{(2}\left(r, \frac{1}{f-a}\right) + \cdots + \bar{N}_{(k}\left(r, \frac{1}{f-a}\right).$$

$$\delta_{k+1}(a, f) = 1 - \overline{\lim_{r\to\infty}} \frac{N_{k+1}\left(r, \frac{1}{f-a}\right)}{T(r, f)} \geq 1 - \overline{\lim_{r\to\infty}} \frac{(k+1)\bar{N}\left(r, \frac{1}{f-a}\right)}{T(r, f)}.$$

$$\delta_{k+1}(0, F) = 1 - \overline{\lim_{r\to\infty}} \frac{N_{k+1}\left(r, \frac{1}{F}\right)}{T(r, F)} \geq 1 - \overline{\lim_{r\to\infty}} \frac{(k+1)\bar{N}\left(r, \frac{1}{F}\right)}{T(r, F)}.$$

Therefore

$$\delta_{k+1}(0, F) = 1 - \overline{\lim_{r\to\infty}} \frac{(k+1)\bar{N}\left(r, \frac{1}{f}\right)}{nT(r, f)} \geq 1 - \frac{(k+1)}{n} = \frac{n-(k+1)}{n}. \tag{4.6}$$

Similarly

$$\delta_{k+1}(0, G) \geq \frac{n-(k+1)}{n}. \tag{4.7}$$

From (4.1)–(4.7), we get

$$\Delta \geq \frac{n-1}{n} + \frac{n-1}{n} + (k+2)\frac{(n-1)}{n} + 2\left[\frac{n-1}{n}\right] + \frac{n-(k+1)}{n} + \frac{n-(k+1)}{n}.$$

Since $n > 3k+8$, we get $\Delta > k+7$.

Considering $F^{(k)}(z) = [f^n(z)]^{(k)}$ and $G^{(k)}(z) = [g^n(z)]^{(k)}$, then by condition of Theorem 2, $F^{(k)}(z)$ and $G^{(k)}(z)$ share the value 1 CM and F and G satisfies conditions of Lemma 5, then by Lemma 5, we deduce that either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$.

Next, we consider two cases.

Case 1:

$$F^{(k)}(z)G^{(k)}(z) \equiv 1; \quad \text{that is } [f^n(z)]^{(k)}[g^n(z)]^{(k)} \equiv 1. \tag{4.8}$$

We prove that

$$f \neq 0, \infty \quad \text{and} \quad g \neq 0, \infty. \tag{4.9}$$

Suppose that $f(z)$ has a zero z_0 of order p, then z_0 is a zero of $[f^n(z)]^{(k)}$ of order $(3k+k_1)p-k = 3pk+k_1p-k$ and z_0 is a pole of $[g^n(z)]^{(k)}$ of order $(3k+k_1)q+k = 3kq+k_1q+k$, where $k_1 > 8$. From (4.8), we get

$$3pk+k_1p-k = 3kq+k_1q+k$$

i.e.

$$3k(p-q)+k_1(p-q) = 2k$$

i.e.

$$(3k+k_1)(p-q) = 2k$$

which is impossible since p and q are integers and $k_1 > 8$.

Therefore

$$f \neq 0 \quad \text{and} \quad g \neq 0.$$

Similarly

$$f \neq \infty \quad \text{and} \quad g \neq \infty.$$

Therefore

$$f \neq 0, \infty \quad \text{and} \quad g \neq 0, \infty. \tag{4.10}$$

From (4.8) and (4.10), we get

$$[f^n(z)]^{(k)} \neq 0 \quad \text{and} \quad [g^n(z)]^{(k)} \neq 0. \tag{4.11}$$

From (4.8), (4.9) and (4.11) and Lemma 4, we get for $k \geq 2$ that $f(z) = c_1 \mathrm{e}^{cz}$, $g(z) = c_2 \mathrm{e}^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1c_2)^n (nc)^{2k} = 1$.

Next, we consider $[f^n(z)]^{(k)}[g^n(z)]^{(k)} \equiv 1$ for the case $k = 1$.

That is

$$n^2 f^{n-1} f' g^{n-1} g' \equiv 1. \tag{4.12}$$

We prove that

$$f \neq 0, \infty \quad \text{and} \quad g \neq 0, \infty. \tag{4.13}$$

In fact, suppose that f has a zero z_0 with order p. Then z_0 is a pole of $g(z)$ (with order q say), by (4.12), we get $(n-1)p + p - 1 = q(n-1) + q + 1$

$$n(p-q) = 2,$$

which is impossible since p and q are integers and $n > 3k+8 = 11$.

Therefore $f \neq 0$ and $g \neq 0$. Similarly $f \neq \infty$ and $g \neq \infty$. Therefore

$$f \neq 0, \infty \quad \text{and} \quad g \neq 0, \infty.$$

Thus there exist two entire functions $\alpha(z)$ and $\beta(z)$ such that

$$f(z) = \mathrm{e}^{\alpha(z)} \quad \text{and} \quad g(z) = \mathrm{e}^{\beta(z)}. \tag{4.14}$$

Inserting these in (4.12), we get

$$n^2 \alpha' \beta' \mathrm{e}^{n(\alpha+\beta)} \equiv 1. \tag{4.15}$$

Thus α' and β' have no zeros and we may set

$$\alpha' = \mathrm{e}^{\delta(z)} \quad \text{and} \quad \beta' = \mathrm{e}^{\gamma(z)}. \tag{4.16}$$

Then (4.15) reduces to

$$n^2 e^{n(\alpha+\beta)+\delta+\gamma} \equiv 1.$$

Differentiating this gives

$$n(\alpha' + \beta') + \delta' + \gamma' \equiv 0;$$

that is

$$n(e^{\delta} + e^{\gamma}) + \delta' + \gamma' \equiv 0.$$

From (4.16), we get

$$n(e^{\delta-\gamma} + 1)e^{0z} + \alpha'' e^{-\delta} + \beta'' e^{-\gamma} \equiv 0.$$

By Lemma 6, we get

$$(e^{\delta-\gamma} + 1) = 0$$

i.e.

$$e^{\delta-\gamma} = -1$$

i.e.

$$\delta - \gamma = (2m+1)\pi i \quad \text{for some integer } m,$$

i.e.

$$\delta = \gamma + (2m+1)\pi i \quad \text{for some integer } m.$$

Inserting this in the above equality, we deduce that $\delta' = \gamma' \equiv 0$, and so δ and γ are constants, i.e.

$$\alpha' \text{ and } \beta' \text{ are constants.} \tag{4.17}$$

From (4.12)–(4.14) and (4.17), we obtain

$$f(z) = c_1 e^{cz}, \qquad g(z) = c_2 e^{-cz}$$

where c_1, c_2 and c are three constants satisfying $(c_1c_2)^n(cn)^2 \equiv -1$.

Therefore for the case 1, i.e. $F^{(k)}G^{(k)} \equiv 1$, for all $k \geq 1$, we get $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$.

Case 2: $F(z) \equiv G(z)$; that is $f^n(z) \equiv g^n(z)$.

This implies $f = tg$, where $t^n = 1$ i.e. t is the nth root of unity.

5. Proof of Theorem 3

From Lemma 3 and (2.1), we have

$$\begin{aligned}
(n+1)T(r,f) &= T\left(r, f^n(f-1)\right) + S(r,f) \\
&\leq \bar{N}\left(r, f^n(f-1)\right) + N\left(r, \frac{1}{f^n(f-1)}\right) \\
&\quad + N\left(r, \frac{1}{[f^n(f-1)]^{(k)} - 1}\right) - N\left(r, \frac{1}{[f^n(f-1)]^{(k+1)}}\right) + S(r,f) \\
&\leq \bar{N}(r,f) + nN\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-1}\right) \\
&\quad + N\left(r, \frac{1}{(f^n(f-1))^{(k)} - 1}\right) - [n-(k+1)]\, N\left(r, \frac{1}{f}\right) + S(r,f) \\
&\leq (k+3)T(r,f) + N\left(r, \frac{1}{[f^n(f-1)]^{(k)} - 1}\right) + S(r,f).
\end{aligned}$$

Therefore

$$(n-k-2)T(r,f) \leq N\left(r, \frac{1}{[f^n(f-1)]^{(k)}-1}\right) + S(r,f). \tag{5.1}$$

Hence, we deduce by (5.1) and $n \geq k+3$ that $[f^n(f-1)]^{(k)}-1$ has infinitely many solutions.

6. Proof of Theorem 4

Let

$$F = f^n(f-1) \quad \text{and} \quad G = g^n(g-1).$$

Consider

$$\begin{aligned}
\Theta(0,F) &= 1 - \overline{\lim_{r\to\infty}} \frac{\bar{N}\left(r,\frac{1}{F}\right)}{T(r,F)} = 1 - \overline{\lim_{r\to\infty}} \frac{\bar{N}\left(r,\frac{1}{f^n(f-1)}\right)}{(n+1)T(r,f)} \\
&= 1 - \overline{\lim_{r\to\infty}} \frac{\bar{N}\left(r,\frac{1}{f}\right) + \bar{N}\left(r,\frac{1}{f-1}\right)}{(n+1)T(r,f)} \geq 1 - \overline{\lim_{r\to\infty}} \frac{2T(r,f)}{(n+1)T(r,f)} \\
&\geq 1 - \frac{2}{n+1} = \frac{n-1}{n+1}.
\end{aligned} \tag{6.1}$$

Similarly

$$\Theta(0,G) \geq \frac{n-1}{n+1}. \tag{6.2}$$

Consider

$$\begin{aligned}
\Theta(\infty,F) &= 1 - \overline{\lim_{r\to\infty}} \frac{\bar{N}(r,F)}{T(r,F)} = 1 - \overline{\lim_{r\to\infty}} \frac{\bar{N}(r,f^n(f-1))}{(n+1)T(r,f)} \\
&= 1 - \overline{\lim_{r\to\infty}} \frac{\bar{N}(r,f)}{(n+1)T(r,f)} \geq 1 - \overline{\lim_{r\to\infty}} \frac{T(r,f)}{(n+1)T(r,f)} \\
&\geq \frac{n}{n+1}.
\end{aligned} \tag{6.3}$$

Similarly

$$\Theta(\infty,G) \geq \frac{n}{n+1}. \tag{6.4}$$

Next, we have

$$N_k\left(r,\frac{1}{F}\right) = \bar{N}\left(r,\frac{1}{F}\right) + \bar{N}_{(2}\left(r,\frac{1}{F}\right) + \cdots + \bar{N}_{(k}\left(r,\frac{1}{F}\right).$$

$$\begin{aligned}
\delta_{k+1}(0,F) &= 1 - \overline{\lim_{r\to\infty}} \frac{N_{k+1}\left(r,\frac{1}{F}\right)}{T(r,F)} = 1 - \overline{\lim_{r\to\infty}} \frac{N_{k+1}\left(r,\frac{1}{f^n(f-1)}\right)}{T(r,F)} \\
&\geq 1 - \overline{\lim_{r\to\infty}} \frac{(k+2)T(r,f)}{(n+1)T(r,f)},
\end{aligned}$$

i.e.

$$\delta_{k+1}(0,F) \geq 1 - \frac{(k+2)}{n+1} = \frac{(n-k-1)}{n+1}. \tag{6.5}$$

Similarly

$$\delta_{k+1}(0,G) \geq \frac{n-k-1}{n+1}. \tag{6.6}$$

We have $\Delta = \Theta(0,F) + \Theta(0,G) + (k+2)\Theta(\infty,F) + 2\Theta(\infty,G) + \delta_{k+1}(0,F) + \delta_{k+1}(0,G)$.

From (6.1)–(6.6), we get

$$\begin{aligned}\Delta &\geq 2\left(\frac{n-1}{n+1}\right) + (k+2)\frac{n}{n+1} + 2\left(\frac{n}{n+1}\right) + \frac{n-k-1}{n+1} + \frac{n-k-1}{n+1}\\ &= \frac{2(n-1)+(k+4)n+2(n-k-1)}{n+1}.\end{aligned}$$

Since $n > 3k+11$, we get $\Delta > k+7$.

Considering $F^{(k)}(z) = [f^n(z)[f(z)-1]]^{(k)}$ and $G^{(k)}(z) = [g^n(z)[g(z)-1]]^{(k)}$, then by the condition of Theorem 4, we obtain that $F^{(k)}$ and $G^{(k)}$ share the value 1 CM and F and G satisfies conditions of Lemma 5, then by Lemma 5, we deduce that either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$.

Next, we consider the case $F^{(k)}G^{(k)} \equiv 1$, that is

$$\left[f^n(z)[f(z)-1]\right]^{(k)}\left[g^n(z)[g(z)-1]\right]^{(k)} \equiv 1. \tag{6.7}$$

Let z_0 be a zero of f of order p. From (6.7) we get z_0 is a pole of g. Suppose that z_0 is a pole of g of order q. Again by (6.7), we obtain

$$np - k = nq + q + k$$

i.e.

$$n(p-q) = q + 2k,$$

which implies that $p \geq q+1$ and $q + 2k \geq n$. Hence

$$p \geq n - 2k + 1. \tag{6.8}$$

Let z_1 be a zero of $f-1$ of order p_1, then z_1 is zero of $[f^n(f-1)]^{(k)}$ of order $p_1 - k$. Therefore from (6.7), we obtain

$$p_1 - k = nq_1 + q_1 + k, \quad \text{since } z_1 \text{ is a pole of } g \text{ of order } q_1$$

i.e.

$$p_1 = (n+1)q_1 + 2k$$

i.e.

$$p_1 \geq n + 2k + 1. \tag{6.9}$$

Let z_2 be a zero of f' of order p_2 that is not a zero of $f(f-1)$, as above, we obtain from (6.7) i.e.

$$\begin{aligned}p_2 - (k-1) &= nq_2 + q_2 + k\\ p_2 &= (n+1)q_2 + 2k - 1\end{aligned}$$

i.e.

$$p_2 \geq n + 2k. \tag{6.10}$$

Moreover, in the same manner as above, we have similar results for the zeros of $[g^n(g-1)]^{(k)}$.

On the other hand, suppose that z_3 is a pole of f. From (6.7), we get that z_3 is the zero of $[g^n(z)[g(z)-1]]^{(k)}$. Thus

$$\bar{N}(r,f) \leq \bar{N}\left(r,\frac{1}{g}\right) + \bar{N}\left(r,\frac{1}{g-1}\right) + \bar{N}\left(r,\frac{1}{g'}\right)$$

$$\leq \frac{1}{n-2k+1} N\left(r, \frac{1}{g}\right) + \frac{1}{n+2k+1} N\left(r, \frac{1}{g-1}\right) + \frac{1}{n+2k} N\left(r, \frac{1}{g'}\right).$$

Since $n \geq 3k+11$, we get

$$\begin{aligned}
\bar{N}(r, f) &\leq \frac{1}{k+12} N\left(r, \frac{1}{g}\right) + \frac{1}{5k+12} N\left(r, \frac{1}{g-1}\right) + \frac{1}{5k+11} N\left(r, \frac{1}{g'}\right) \\
&\leq \frac{1}{13} N\left(r, \frac{1}{g}\right) + \frac{1}{17} N\left(r, \frac{1}{g-1}\right) + \frac{2}{16} N\left(r, \frac{1}{g'}\right) \\
&\leq \left(\frac{1}{13} + \frac{1}{17} + \frac{1}{8}\right) T(r, g) + S(r, g) \\
&\leq (0.261) T(r, g) + S(r, g). \qquad (6.11)
\end{aligned}$$

From Lemma 2 and from (6.11), we get

$$\begin{aligned}
T(r, f) &\leq \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{f-1}\right) + \bar{N}(r, f) + S(r, f) \\
&\leq \frac{1}{13} N\left(r, \frac{1}{f}\right) + \frac{1}{17} N\left(r, \frac{1}{f-1}\right) + (0.261) T(r, g) + S(r, f) + S(r, g) \\
&\leq (0.1358) T(r, f) + (0.261) T(r, g) + S(r, f) + S(r, g). \qquad (6.12)
\end{aligned}$$

Similarly, we have

$$T(r, g) \leq (0.1358) T(r, g) + (0.261) T(r, f) + S(r, f) + S(r, g). \qquad (6.13)$$

Adding (6.12) and (6.13), we obtain

$$T(r, f) + T(r, g) \leq 0.7936\,(T(r, f) + T(r, g)) + S(r, f) + S(r, g).$$

i.e.

$$(0.2064)\,[T(r, f) + T(r, g)] \leq S(r, f) + S(r, g),$$

which is a contradiction.

Case 2: If $F \equiv G$, that is

$$f^n(f-1) = g^n(g-1). \qquad (6.14)$$

Suppose $f \not\equiv g$, then we consider two cases:

(i) Let $h = \frac{f}{g}$ be a constant. Then from (6.14) it follows that $h \neq 1$, $h^n \neq 1$, $h^{n+1} \neq 1$ and $g = \frac{1-h^n}{1-h^{n+1}} =$ constant, which leads to a contradiction.

(ii) Let $h = \frac{f}{g}$ be not a constant. Since $f \not\equiv g$, we have $h \not\equiv 1$ and hence we deduce that

$$g = \frac{1-h^n}{1-h^{n+1}} \quad \text{and} \quad f = \left(\frac{1-h^n}{1-h^{n+1}}\right) h = \frac{\left(1+h+h^2+\cdots+h^{n-1}\right) h}{1+h+h^2+h^n},$$

where h is a non-constant meromorphic function. It follows that

$$T(r, f) = T(r, gh) = (n+1) T(r, h) + S(r, f).$$

On the other hand, by the second fundamental theorem, we deduce

$$\bar{N}(r, f) = \sum_{j=1}^{n} \bar{N}\left(r, \frac{1}{h-\alpha_j}\right) \geq (n-2) T(r, h) + S(r, f),$$

where $\alpha_j (\neq 1)$ $(j = 1, 2 \ldots n)$ are distinct roots of the algebraic equation $h^{n+1} = 1$.

We have

$$\begin{aligned}\Theta(\infty, f) &= 1 - \overline{\lim_{r\to\infty}} \frac{\bar{N}(r, f)}{T(r, f)} \leq 1 - \overline{\lim_{r\to\infty}} \frac{(n-2)T(r, h) + S(r, f)}{T(r, f)} \\ &\leq 1 - \overline{\lim_{r\to\infty}} \frac{(n-2)T(r, h) + S(r, f)}{(n+1)T(r, h) + S(r, f)} \leq 1 - \frac{(n-2)}{(n+1)} = \frac{3}{n+1}.\end{aligned}$$

i.e. $\Theta(\infty, f) \leq \frac{3}{n+1}$, which contradicts the assumption $\Theta(\infty, f) > \frac{3}{n+1}$.

Thus $f \equiv g$. This completes the proof of Theorem 4.

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